NMOS SPECIAL ROUND 2019 SOLUTION

1 $36\times\left(\frac{1}{1\times 6}+\frac{1}{6\times 11}+\frac{1}{11\times 16}+\frac{1}{16\times 21}+\frac{1}{21\times 26}+\frac{1}{26\times 31}+\frac{1}{31\times 36}\right)$

$=36\times\left[\frac{1}{5}\times\left(1-\frac{1}{6}+\frac{1}{6}-\frac{1}{11}+\cdots+\frac{1}{31}-\frac{1}{36}\right)\right]$

$=36\times\frac{1}{5}\times\frac{35}{36}=7$

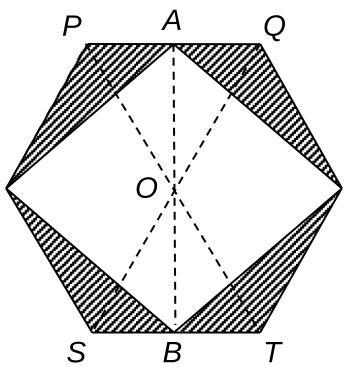
2 $\langle A|B\rangle$ is essentially a product of numbers. The last digit of a product can be deduced from the product of the last digits of each number.

Since $\langle 2006|2015\rangle$ contains 2010, its last digit is 0.

$\langle 2016|2019\rangle \equiv 6\times 7\times 8\times 9 \equiv 4\ (\text{mod}10)$.

Hence, the last digit of the sum $\langle 2006|2015\rangle+\langle 2016|2019\rangle$ is 4.

3



Join AB, PT and SQ. The total area of the shaded region equals to the sum of the area of triangle OPQ and the area of triangle OST, which is

$\frac{96}{6}\times 2=32\,\text{cm}^2$.

4 By the question,

	A	:	B	:	C	:	D	:	E	Total
At first	1	:	2	:	3	:	4	:	5	15
Actual day	6	:	7	:	8	:	9	:	10	40

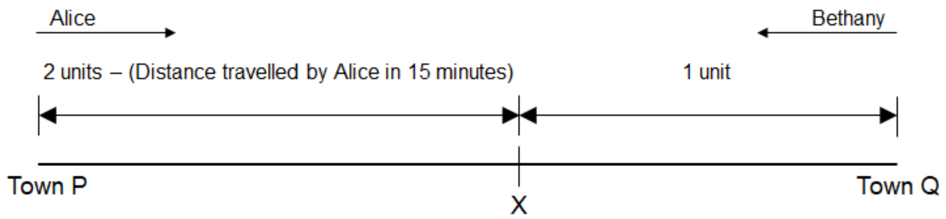
Since the total number of gifts remains the same, we have

	A	:	B	:	C	:	D	:	E	Total
At first	$1(\times 8)$	:	$2(\times 8)$	:	$3(\times 8)$	:	$4(\times 8)$	:	$5(\times 8)$	$15(\times 8)$
Actual day	$6(\times 3)$	:	$7(\times 3)$	:	$8(\times 3)$	:	$9(\times 3)$	:	$10(\times 3)$	$40(\times 3)$

Since the number of gifts that each class received is a whole number, Class *D* should be the class that received 5 less gifts.

Hence, the total number of Children's Day gifts is $\frac{5}{32-27}\times 120 = 120$.

5 Let X be the point at which Alice meets Bethany.



For the entire journey, Alice travels for $15+30=45$ minutes less compared to Bethany. Hence, Bethany takes $45\times 2=90$ minutes for the entire journey.

Let 1 unit be the distance between X and Town Q. Hence, Bethany would have travelled (3 units $-$ Distance travelled by Alice in 15 minutes) during these 90 minutes.

Since Bethany will need to spend 30 minutes to cover the same distance as Alice would in 15 minutes, so in 90 minutes, Bethany would have travelled

(3 units $-$ Distance travelled by Bethany in 30 minutes).

In 120 minutes, Bethany would have travelled 3 units.

Therefore, time taken for Bethany to travel 1 unit is 40 minutes.

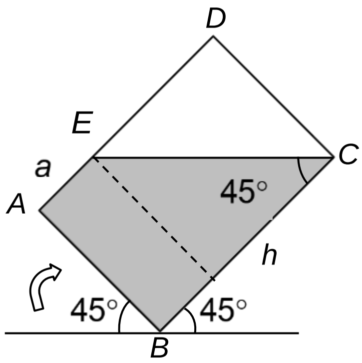
6 Note that $\frac{1}{2019} < \frac{1}{2018} < \frac{1}{2017}$, hence, $\frac{6}{2019} < \frac{2}{2017} + \frac{2}{2018} + \frac{2}{2019} < \frac{6}{2017}$.

$$336\frac{1}{6} = \frac{2017}{6} < \frac{1}{\frac{2}{2017} + \frac{2}{2018} + \frac{2}{2019}} < \frac{2019}{6} = 336\frac{1}{2}.$$

Therefore, $P = 336$.

7 Let the length of BC be h and the length of AE be a.

The cross sectional area of the water in Figure 1 is $60 \times \left(\frac{3}{4}h\right) = 45h$.



In Figure 2, the cross section of the water is a trapezium.

The area is $\frac{1}{2} \times (a + h) \times 60 = 30(h + a)$.

Since $h - a = 60$, hence, $30(h + h - 60) = 45h \Rightarrow h = 120$ cm.

8 Let t be the time taken (in minutes) for Gopal on the first day.

Distance travelled by Gopal on the first day $= \frac{49t}{60}$ km.

Distance travelled by Gopal on the second day $= \frac{45(t+8)}{60}$ km.

$$\frac{45(t+8)}{60} = \frac{49t}{60} + 1$$

$$45(t+8) = 49t + 60$$

$$45t + 360 = 49t + 60$$

$$4t = 300$$

$$t = 75$$

9 Since $\frac{A+23}{A-37} = 1 + \frac{60}{A-37}$ is a positive whole number, $(A-37)$ should be a positive factor of 60. There are 12 positive factors of 60.

Hence, there are 12 different possible values of A.

10 We claim that at least 8 cross shapes are needed.

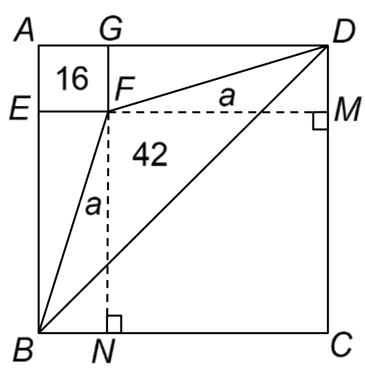
Notice that each 4×4 chessboard could be covered by four T-shapes, and hence, four cross shapes. Now we duplicate this covering scheme for a 4×8 chessboard.

1			
1	1		3
1	2	3	3
2	2	2	3

If we use 7 or less cross shapes, there are at most $7 \times 5 = 35$ squares and by covering all 32 squares on the chessboard, there are at most 3 squares for overlapping or extrusion. Hence, at least one of the border lines has neither overlapping nor extrusion. This is impossible.

11 Let FM be perpendicular to CD and intersect CD at M.

Let FN be perpendicular to BC and intersect BC at N.



Since both $ABCD$ and $AEFG$ are squares, hence, $FNCM$ is also a square.

Let the length of each side of $FNCM$ be a, then the length of each side of $ABCD$ is $(a+4)$. Area of triangle $ABD = \frac{1}{2}(a+4)^2$.

Also, area of triangle ABD

$= $ area of square $AEFG + 2\times$area of $FGD +$ area of FBD

$= 16 + 4a + 42$.

Hence, $\frac{1}{2}(a+4)^2 = 16 + 4a + 42 \Rightarrow a^2 = 100 \Rightarrow a = 10$.

Thus, each side of square $ABCD$ is 14 cm, and the area of square $ABCD$ is 196 cm^2.

12 Let $A+B=7m$ and $A^2+B^2=7n$, whereby m and n are positive whole numbers.

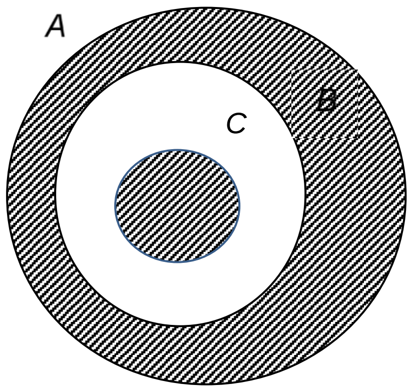
Since $2AB=(A+B)^2-(A^2+B^2)=7(7m^2-n)$, hence, at least one of A and B is a multiple of 7.

Since $A+B=7m$, hence, both A and B are multiples of 7.

In order to maximize the difference $B-A$, we have $B=9996$ (the largest 4-digit multiple of 7) and $A=14$ (the smallest 2-digit multiple of 7).

Thus, the largest possible value of $B-A$ is $9996-14=9982$.

13 Let $A=$\{even numbers from 1 to 2019\}, $B=$\{multiples of 4 from 1 to 2019\}, and $C=$\{multiples of 8 from 1 to 2019\}.



The Venn Diagram is as shown.

Total number of requested even numbers

$= |A-B| + |C|$

$= \left\lfloor \frac{2019}{2} \right\rfloor - \left\lfloor \frac{2019}{4} \right\rfloor + \left\lfloor \frac{2019}{8} \right\rfloor = 757.$

14 There are 4 cases to pave the block.

Case 1. All 6 tiles are placed horizontally. There is only 1 way to pave.

Case 2. Only 4 tiles are placed horizontally, and the other 2 tiles are placed vertically. There are $3+2+1=6$ ways to pave.

Case 3. Only 2 tiles are placed horizontally, and the other 4 tiles are placed vertically. There are $3+2=5$ ways to pave.

Case 4. All 6 tiles are placed vertically. There is only 1 way to pave.

Hence, there are altogether 13 ways to pave the block.

15 Let the requested 3-digit positive whole number be $\overline{abc}$, whose value is equal to the sum $a!+b!+c!$.

Note that $1!=1, 2!=2, 3!=6, 4!=24, 5!=120$ and $6!=720$. None of the three digits can be 7, 8 or 9, as its factorial value will exceed 1000.

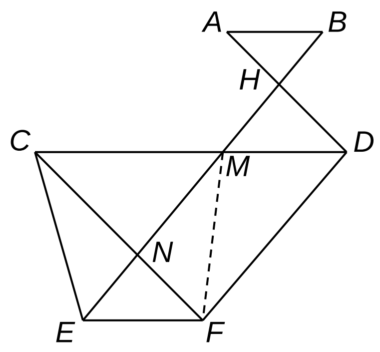
If 6 is one of the three digits, then the sum of the three factorials will be greater than 720. However, the first digit a cannot be 7, 8 or 9. Hence, none of the three digits can be 6.

On the other hand, since $4!+4!+4!=72$ is the largest sum we can get using only the digits $1,2,3$ or 4, hence, in order to have a 3-digit sum, at least one digit must be 5.

$5!+5!+5!=360$ implies that the digits cannot be all 5. Similarly, it could be checked that numbers like $\overline{a55}, \overline{5b5}$ and $\overline{55c}$ (where the missing digit is 1, 2, 3 or 4) are not possible. Hence, only one digit is 5.

As the largest possible 3-digit number is $5!+4!+4!=168$, $a=1$. Hence, either $b=5$ or $c=5$. Checking through all possible numbers as $\overline{1b5}$ and $\overline{15c}$, we find that there is only one possible 3-digit number 145, as $1!+4!+5!=145$.

16



Suppose the area of triangle AHB is 1 unit^2.

Since AB is parallel to MD, and $\frac{MD}{AB} = 2$, hence, the area of triangle MHD is 4 units^2. Since AD is parallel to CF, and $\frac{CM}{MD} = \frac{3}{2}$, hence, the area of triangle CNM is 9 units^2.

Since $\frac{\text{Area of } \Delta ENF}{\text{Area of } \Delta ENC} = \frac{\text{Area of } \Delta ENC}{\text{Area of } \Delta CNM} = \frac{\text{Area of } \Delta FNM}{\text{Area of } \Delta CNM} = \frac{2}{3}$, hence, the area of triangle ENC is 6 units^2, and the area of triangle FNM is 6 units^2, and the area of triangle ENF is 4 units^2.

Since $\frac{\text{Area of } \Delta FMD}{\text{Area of } \Delta FMC} = \frac{MD}{MC} = \frac{2}{3}$, hence, the area of triangle FMD is 10 units^2.

Therefore, the total area of quadrilateral $CDFE$ is 35 units^2, and $m = 35$.

17 Obviously John could not have tossed 2019 Heads.

Notice that after an odd number of Tails, the numerator of the fraction on the board would contain a factor 5, and only after an even number of Tails, would this factor 5 come back to the denominator. Hence, we conclude that John must have tossed an odd number of Heads, and the largest possible number of Heads is 2017. This is possible via the following.

(1) John first tossed 1008 Heads, with $\frac{4}{5}$ doubled 1008 times.

(2) Next, John tossed a Tail, and the number on the board became $\frac{5}{4}$ being halved 1008 times.

(3) John would then toss 1009 Heads, and the number on the board would be doubled 1009 times, which gave $\frac{5}{2}$.

(4) One more Tail led to $\frac{2}{5}$.

18 Consider the 8-digit number *ddmmyyyy*. Since the first digit is unchanged, it is always among $0,1,2$ and 3. Call it k.

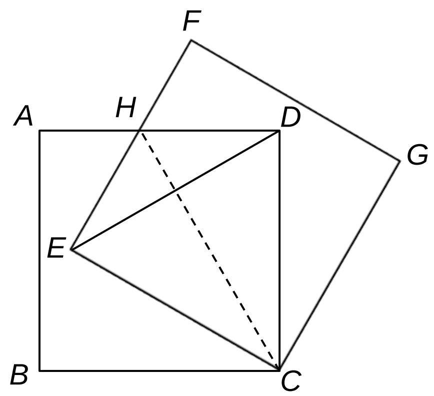
Notice that the sum of the month digits is at most 9, and the sum of the year digits is at most $9+9+9+9=36$. Hence, after the first use of the crystal ball, the new number is at most $\overline{k54}$, where $9+9+36=54$.

After the second use of the crystal ball, the new number is at most $\overline{k13}$, because the sum of digits for those not exceeding 54 is at most 13.

After the third use of the crystal ball, the final number must be a two-digit number $\overline{k_1k_2}$, where $k_1=0,1,2,3$ and k_2 can be any non-zero digit. Indeed, k_2 is the remainder when the sum of the digits is divided by 9.

In conclusion, there are $4\times 9=36$ possible magic codes.

19



The length of each side of two squares is 10 cm.

Area of $ABCH=\frac{1}{2}\times(AH+10)\times 10=\frac{125}{2}$ cm^2. Hence, $AH=2.5$ cm.

Join CH. $CH=\sqrt{AB^2+(BC-AH)^2}=12.5$ cm.

Since triangle CED is an isosceles triangle, $\angle CED=\angle CDE$.

$\angle HED = 90° - \angle CED = 90° - \angle CDE = \angle HDE$, which implies that triangle HED is also an isosceles triangle. Hence, CH is perpendicular to DE.

Area of quadrilateral $HECD = \frac{1}{2} \times DE \times HC = 100 \times 2 - 125 = 75$ cm^2.

Therefore, $DE = 12$ cm.

20 Since $\frac{d}{e} \times f$ is a multiplication, in order to maximize N, we should maximize $\frac{d}{e} \times f$. Hence, $e = 1$, and d and f would be 8 and 9. ($(d, f) = (8,9)$ or $(d, f) = (9,8)$). The maximum value of $\frac{d}{e} \times f$ is 72.

For $a + \frac{b}{c} - \left(g + \frac{h}{j}\right) = (a-g) + \left(\frac{b}{c} - \frac{h}{j}\right)$, since neither c or j is 1, in order to maximize $(a-g) + \left(\frac{b}{c} - \frac{h}{j}\right)$, we should maximize $(a-g)$. The maximum value of $(a-g)$ is $7 - 2 = 5$.

Then b, c, h and j would be $3, 4, 5$ and 6. In order to maximize $\frac{b}{c} - \frac{h}{j} = \frac{b \times j - c \times h}{c \times j}$, we should minimize $c \times j$. The minimum value of $c \times j$ is $3 \times 4 = 12$.

If $c = 3$, $j = 4$, $b = 5$ and $h = 6$, then $\frac{b}{c} - \frac{h}{j} = \frac{1}{6}$.

If $c = 3$, $j = 4$, $b = 6$ and $h = 5$, then $\frac{b}{c} - \frac{h}{j} = \frac{3}{4}$.

If $c = 4$, $j = 3$, $b = 5$ and $h = 6$, then $\frac{b}{c} - \frac{h}{j} = -\frac{3}{4}$.

If $c = 4$, $j = 3$, $b = 6$ and $h = 5$, then $\frac{b}{c} - \frac{h}{j} = -\frac{1}{6}$.

If $c = 3$, $j = 5$, $b = 4$ and $h = 6$, then $\frac{b}{c} - \frac{h}{j} = \frac{2}{15}$.

If $c=3$, $j=5$, $b=6$ and $h=4$, then $\frac{b}{c}-\frac{h}{j}=\frac{6}{5}$.

If $c=5$, $j=3$, $b=4$ and $h=6$, then $\frac{b}{c}-\frac{h}{j}=-\frac{6}{5}$.

If $c=5$, $j=3$, $b=6$ and $h=4$, then $\frac{b}{c}-\frac{h}{j}=-\frac{2}{15}$.

If $c=3$, $j=6$, $b=4$ and $h=5$, then $\frac{b}{c}-\frac{h}{j}=\frac{1}{2}$.

If $c=3$, $j=6$, $b=5$ and $h=4$, then $\frac{b}{c}-\frac{h}{j}=1$.

If $c=6$, $j=3$, $b=5$ and $h=4$, then $\frac{b}{c}-\frac{h}{j}=-\frac{1}{2}$.

If $c=6$, $j=3$, $b=4$ and $h=5$, then $\frac{b}{c}-\frac{h}{j}=-1$.

Hence, the maximum value of $\frac{b}{c}-\frac{h}{j}$ is 1.

Therefore, the maximum value of the result N is $72+5+1=78$.