SOLUTION

1. Simplify the multiplication as follows:

$$2000\times\left(1-\frac{1}{2}\right)\times\left(1-\frac{1}{3}\right)\times\left(1-\frac{1}{4}\right)\times\cdots\times\left(1-\frac{1}{100}\right)$$

$$=2000\times\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\times\cdots\times\frac{99}{100}=2000\times\frac{1}{100}=20$$

2. Since $2022=2\times3\times337$, $d(2022)=8$.

One sees that $X=102$ and $Y=1001$, i.e., $X+Y=1103$.

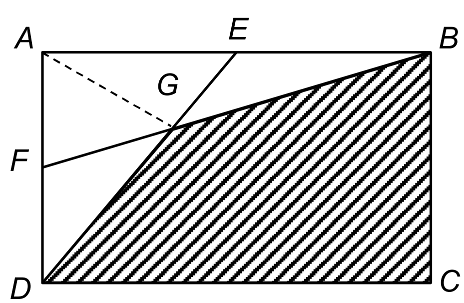
3. Since if Benny's age is doubled and Calvin's age is halved, both will be of the same age, Benny's current age : Calvin's current age $=1:4$.

Let Benny's current age be 1unit, then Andy's current age is (2 units-7) and Calvin's current age is 4units. Hence,

$2u-7+1u+4u=42\times3=126\Rightarrow7u=133\Rightarrow1u=19$

Thus, Calvin's current age is $4\times19=76$ years old.

4.



Join AG. Let $[ABCD]$ denote the area of the rectangle $ABCD$.

Since $AE=EB$ and $AF=FD$, let $[AEG]=[BEG]=m$ and $[AFG]=[DFG]=n$.

$[AFB]=\frac{1}{4}[ABCD]=[AED]\Rightarrow m+2n=n+2m\Rightarrow m=n$.

Hence, area of non-shaded region is $\frac{4}{3}\times\frac{1}{2}\times(5\times6)=20\,\text{cm}^2$.

The area of shaded region is $(3+3)\times(5+5)-20=40\,\text{cm}^2$.

5.

Rest Days / Work	Sat, Sun	Work	Tue, Wed	Work	Fri, Sat	Work	Mon, Tue	Work	Thu, Fri	Work	Sun, Mon
No. of days	**-**	**8**	**2**	**8**	**2**	**8**	**2**	**8**	**2**	**8**	**-**

The table above shows the number of days that must elapse before the next rest day falls on Sunday again.

Number of days = $8\times5+2\times4=48$. This translates to $48\div7=6$ remainder 6 weeks. Thus, 7 weeks must elapse before the next rest day falls on Sunday again.

6. Since $A+17$ is a multiple of 5, A has remainder 3 when it is divided by 5.
Since $A-17$ is a multiple of 6, A has remainder 5 when it is divided by 6.
Let $A=5m+3$, where m is an integer.
$5m+3\equiv5\ (\text{mod}\,6)\Rightarrow m\equiv4\ (\text{mod}\,6)$
Hence, $A=5m+3=5(6k+4)+3=30k+23$, where k is an integer.
Thus, the largest possible value of A is $30\times2+23=83$.

7. Area of the shaded region is $\frac{1}{2}\times2\times3=3$ units.

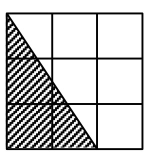


Figure 1

Case 1: Triangles of base 2 units and height 3 units, as shown in Figure 1. There are $4\times2=8$ such triangles.

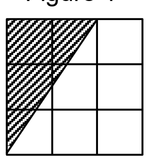


Figure 2

Case 2: Triangles of base 2 units and height 3 units, as shown in Figure 2. There are $4\times2=8$ such triangles.

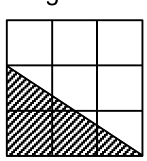


Figure 3

Case 3: Triangles of base 3 units and height 2 units, as shown in Figure 3. There are $4\times2=8$ such triangles.

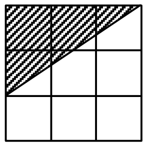


Figure 4

Case 4: Triangles of base 3 units and height 2 units, as shown in Figure 4. There are $4 \times 2 = 8$ such triangles.

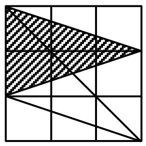


Figure 5

Case 5: Triangles of base 2 units and height 3 units, as shown in Figure 5. There are $2 \times 2 \times 2 = 8$ such triangles.

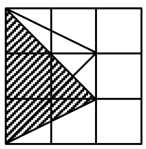


Figure 6

Case 6: Triangles of base 3 units and height 2 units, as shown in Figure 6. There are $2 \times 2 \times 2 = 8$ such triangles.

There are altogether $8+8+8+8+8+8=48$ triangles.

8. Notice that $a * b = \frac{7a-b}{7a+b} = 1$ if and only if $b = 0$. Hence, we conclude that $n^2 * n^3 = 0$.

One also sees that $a * b = \frac{7a-b}{7a+b} = 0$ if and only if $b = 7a$, i.e., $n^3 = 7n^2$.

We conclude that $n = 7$ is the only positive integer.

9. Suppose one machine from Warehouse B is sent to Factory I.

Consider the cost of (B,I) and (A,II), i.e., sending one machine from Warehouse B to Factory I and one machine from Warehouse A to Factory II. It costs S$135.

If we switch the job, notice that (B,II) and (A,I) only costs S$120, a cheaper option. Hence, we conclude that all 7 machines in Warehouse B should be sent to Factory II. The total cost is $50 \times 9 + 100 \times 3 + 70 \times 7 = 1240$.

10. Notice that $\overline{ab} > \left(P\left(\overline{ab}\right)\right)^2$ is only possible when $P\left(\overline{ab}\right) \le 9$.

$P\left(\overline{ab}\right)$	$\overline{ab} \le (ab)^2$	Accepted	Count
9	19, 33	91	1
8	18, 24, 42	81	1
7	17	71	1
6	16, 23, 32	61	1
5	15	51	1
4	14	41, 22	2
3		13, 31	2
2		12, 21	2
1		11	1
0		10, 20, …, 90	9

There are 21 such two-digit positive integers.

11. During the first 10 minutes, car X travelled $2.5 \times \frac{1}{6} = \frac{5}{12}$ km longer than car Y.

During the last 25 minutes, car X travelled $0.5 \times \frac{25}{60} = \frac{5}{24}$ km longer than car Y.

As the two cars reached station B at the same time, during the in-between 5 minutes, car Y should travel $\frac{5}{12} + \frac{5}{24} = \frac{5}{8}$ km longer than car X, whereby car Y should be faster than car X by $\frac{5}{8} \div \frac{5}{60} = 7.5$ km per hour.

Hence, car X decreased speed by $7.5 + 2.5 = 10$ km per hour.

12. If $(M-13)^2 + 25 - M = M + 2$, we have $(M-13)^2 = 2M - 23$. Now $2M - 23$ is non-negative and we must have $M \ge 12$. Check that $M = 12$ is a solution.

If $(M-13)^2 + 25 - M = M - 2$, we have $(M-13)^2 = 2M - 27$. Similarly, one sees that $M = 14$ is a solution.

We conclude that $M = 12$ is the smallest positive integer satisfying the conditions.

13. If the fire station is located along the rhombus, it will travel at least 2 AB distance, i.e., half of the perimeter of the square, to reach the midpoint of BC. This is also the case if the fire station is located along the square: at least half of the perimeter of the square to reach the position which is symmetric about the centre of the square.

In conclusion, $\frac{9}{60}p \geq 2AB = 14$ and the smallest possible value of p is 94. One may set up the fire station at point E.

14. It is easy to see that $D = 9$. Refer to the fifth row where $(\overline{*7}) \times (*) = \overline{**}$. There are only a few possibilities: $17 \times 4 = 68$, $27 \times 2 = 54$, or $37, 47, 57, 67$ multiplied by 1.

If the divisor is 17, then the last step gives $17 \times 7 = 119$. Now $C = 9 = D$, which is not allowed.

One may also check that the divisor cannot be $37, 47, 57, 67$. The only answer is $27 \times 527 = 14229$. Hence, $A + B + C + D = 16$.

15. $2020 \div 5 = 404$

From the first circle, the students who stayed behind are

$5 \times 1,\ 5 \times 2,\ 5 \times 3, \ldots,\ 5 \times 404$. They will proceed to form the second circle.

$404 \div 5 = 80$ remainder 4

From the second circle, the students who stayed behind are

$5 \times 5 \times 1,\ 5 \times 5 \times 2, \ldots,\ 5 \times 5 \times 80$. They will proceed to form the third circle.

$80 \div 5 = 16$

From the third circle, the students who stayed behind are

$5 \times 5 \times 5 \times 1,\ 5 \times 5 \times 5 \times 2, \ldots,\ 5 \times 5 \times 5 \times 16$. They will proceed to form the fourth circle.

$16 \div 5 = 3$ remainder 1

From the fourth circle, the students who stayed behind are

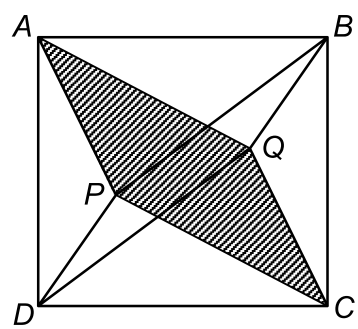
$5 \times 5 \times 5 \times 5 \times 1,\ 5 \times 5 \times 5 \times 5 \times 2,\ 5 \times 5 \times 5 \times 5 \times 3$

Thus, the last student who stays in the smallest circle is $5 \times 5 \times 5 \times 5 \times 3 = 1875$

16. Let the river flow speed be m (km/h) and the speed of the slow ship upstream be n (km/h). Now the speed of the fast ship in still water is $n+m$ (km/h).
According to the schedule, the slow ship moves upstream with speed n (km/h) and travels 90 km. Meanwhile, the fast ship moves downstream with speed $n+2m$ (km/h) and travels 180 km.
It follows that $(n+2m)$ is twice of $n-m$. Hence, n is equal to $4m$.

In reality, the slow ship travels 135 km. The fast ship travels with speed $n+2m=6m$ (km/h) for 2 hours, and then the rest of the time with speed m (km/h). Since the total time is $\frac{135}{n-m}=\frac{135}{3m}$, the fast ship is carried forward by the river flow for $\frac{135}{3m}-2$ hours.
Notice that the fast ship also travels 135 km. We must have: $2(6m)+\left(\frac{135}{3m}-2\right)m=270-135=135$, i.e., $12m+45-2m=135$.
It follows that $m=9$ km per hour.

17.



Since $[ABP]+[DPC]=\frac{1}{2}\times[ABCD]$, hence, $\frac{[DPC]}{[ABCD]}=\frac{1}{2}\times\frac{2}{3+2}=\frac{1}{5}$.
Since $[ADP]+[BCP]=\frac{1}{2}\times[ABCD]$, hence, $\frac{[ADP]}{[ABCD]}=\frac{1}{2}\times\frac{3}{3+7}=\frac{3}{20}$.
Since $[ABQ]+[CDQ]=\frac{1}{2}\times[ABCD]$, hence, $\frac{[ABQ]}{[ABCD]}=\frac{1}{2}\times\frac{3}{3+5}=\frac{3}{16}$.
Since $[ADQ]+[BCQ]=\frac{1}{2}\times[ABCD]$, hence, $\frac{[BCQ]}{[ABCD]}=\frac{1}{2}\times\frac{1}{4+1}=\frac{1}{10}$.

Hence, $\frac{[APCQ]}{[ABCD]} = 1 - \frac{1}{5} - \frac{3}{20} - \frac{3}{16} - \frac{1}{10} = 1 - \left(\frac{16+12+15+8}{80}\right) = 1 - \frac{51}{80} = \frac{29}{80}$.
Thus $m + n = 29 + 80 = 109$.

18. Since $A + 35 + 89 = 124 + A$,

(i) the number below 1 is $35 + 89 - 1 = 123$;

(ii) the number next to 1 is $(A + 124) - 123 - 89 = A - 88$;

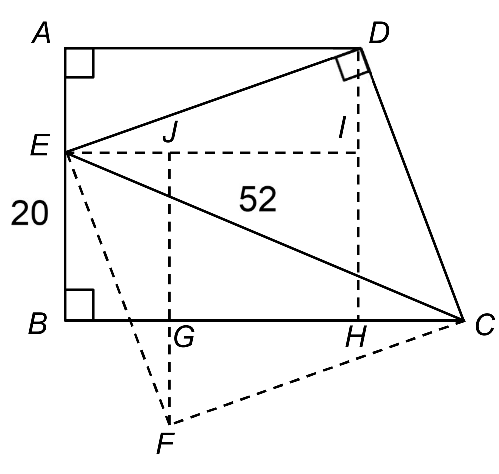
(iii) the number below 89 is $(A + 124) - (A - 88) - 1 = 211$;

(iv) the number below 211 is $(A + 124) - 89 - 211 = A - 176$.

The sum of numbers on the diagonal is $A + (A - 88) + (A - 176)$, hence, $A + (A - 88) + (A - 176) = A + 124$. Therefore, $A = 194$.

19. For any three-digit number, the smallest sum of digits is 1 (for 100 only), and the largest sum of digits is 27 (for 999 only). The other possible sums of digits are 2, 3, 4, …, 26, and each will have at least 3 three-digit numbers. Suppose we have 27 pigeonholes as $\{1\}, \{2\}, \{3\}, \cdots, \{25\}, \{26\}, \{27\}$, the least value of n in order to get at least 3 numbers whose sum of digits are the same is $2 \times 25 + 2 + 1 = 53$.

20.



Area of ΔADE is $\frac{1}{4} \times \left(\frac{52^2}{2} - 20^2\right) = 238$ cm^2.

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