Singapore International Mathematics Challenge 2018

Experior. Expono. Excedo.

# THEME: CITY LIFE

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Cities are one of the central features of modern human civilisation. There is an 'internal' aspect to a city: how information travels around inside it, and how people move about and interact. And there is also an 'external' aspect: how the city evolves and grows with time.

The aim of the SIMC 2018 is to focus on some of these phenomena. There is a problem about spy networks in a city: how information spreads in the network, and how this changes when we look at the finer subdivision of the city into its districts. There is a problem about the movement and interaction of people: when two people walking in opposite directions in a confined environment are going to collide, how does their strategy of conflict resolution affect their later collisions? And there is a problem about city growth: how does the way in which the city 'swallows up' land near it affect its growth, and will the city eventually reach a steady size or not?

## About the Challenge:

There are three sections to the Challenge – Section A, B and C. The three sections carry equal mark with each section consists of 3 to 4 questions. The weighting for each question indicates its contribution to the final score. It also serves as a guide on the amount of time and effort your team should spend on the question. Different teams may find different sections easy and hard. Each section also has an open-ended section, where you are invited not to solve a specific question but rather to invent new models and analyse them.

## Written Report Requirements:

1. Your report should not exceed 10 pages. It will be read by judges. Present the main results and ideas only. You may omit details such as the steps of computations. However, you should be able to explain the details of your work during the oral presentations, if requested by the judges.

2. Teams should acknowledge all sources used.

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3. Your report must:
    - Be saved in PDF format.
    - Be single spacing with font size 12. Do not try to squeeze in the details by reducing the spacing between lines, or the margins.
    - Include your school name in FULL, on the first page of your report.

## Oral Presentations:

1. Each team has to present 3 times. Each presentation is 20 minutes and will be followed by a 10-minute Question & Answer session.
2. During each presentation, you are expected to showcase the most interesting/challenging part(s) of each of the three sections A, B and C.
3. During each Question & Answer session, you may be asked to clarify certain parts of your presentation or certain steps in your written report.

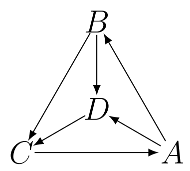
**There are 3 printed pages (excluding the pink cover page)**

# Section A: Master Spies

A city contains a network of $n$ spies. When a spy receives a piece of information, he tells everyone he can, and then they tell everyone they can, and so on. For security reasons, communication between any two given spies is one-way only, but apart from that communication should be as quick as possible, so for any two spies exactly one can send messages to the other.

A spy $A$ is called a 'master spy' if for any other spy $B$, either $A$ can send messages directly to $B$ or else there is a spy $C$ such that $A$ can send messages to $C$ and $C$ can send messages to $B$. In other words, thinking of messages as being sent once per day, information known by spy $A$ will become known by all spies after at most two days.

For example, if four spies can send messages to each other in the direction of the arrows shown in the figure below, then there are three master spies: $A$, $B$ and $C$.




**Question 1a.** Show that, in any spy network, there is a master spy. [*5 points*]

**Question 1b.** Show that there cannot be exactly two master spies. [*5 points*]

**Question 1c.** For which $n$ does there exist a spy network in which every spy is a master spy? [*5 points*]

Now suppose that the city is divided into two districts. For added security, two spies from the same district never communicate directly; but as before for any two spies in different districts exactly one can send messages to the other. If there is more than one spy with the property that he never receives messages then obviously information can never be sent from one spy throughout the network, so we we will always suppose that there is at most one spy with this property.

**Question 2a.** Give an example to show that now there need not be any master spy. Further, give an example to show that there need not even be any spy whose information will reach everyone within three days. [*5 points*]

**Question 2b.** Show that there is now always a 'semi-master' spy $A$, meaning that information known by $A$ will become known by all spies after at most four days. [*5 points*]

**Question 2c.** What happens if we have more districts? [*5 points*]

**Question 3.** How could you modify the model to make it more realistic? What results can you then prove? [*10 points*]

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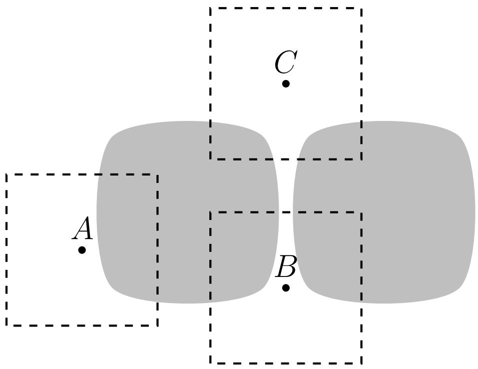
# Section B: City growth

A model for the growth of a city is as follows. The city starts as a bounded region in the plane. We fix a constant $0 < c < 1$, called the 'habitation coefficient'. For any point in the plane, its 'neighbourhood' is the square with sides that are parallel to the axes and have length 1, centred at that point. If the city covers a proportion $c$ or greater of the neighbourhood of a point, then that point becomes part of the city after a year.

For example, in the figure below, the city covers the gray regions and the squares indicate the neighbourhoods of points $A$, $B$ and $C$. If the habitation coefficient $c$ is $1/4$, then $A$ and $B$ join the city next year since their neighbourhoods already contain an area larger than $c$ that is part of the city, but $C$ does not. However, $C$ will join the city in a later year once more points in its neighbourhood have joined the city first. (Note that there is no requirement that the initial region forms a connected shape; the only condition is that it is bounded.)




The initial region covered by the city is called 'successful' if every point of the plane will eventually become part of the city.

**Question 1a.** Prove that no initial region is successful if $c \geq 1/2$. *[5 points]*

**Question 1b.** Conversely, show that for any $c < 1/2$ a successful initial region does exist. *[5 points]*

We say that an initial region is 'optimal' if it is successful and its area is exactly $c$.

**Question 2a.** Find a $0 < c < 1/2$ for which an optimal initial region exists. *[5 points]*

**Question 2b.** Find a $0 < c < 1/2$ for which no initial region is optimal. *[10 points]*

**Question 2c.** What more can you say about for which $0 < c < 1/2$ there exists an optimal initial region? (A complete answer is not expected.) *[5 points]*

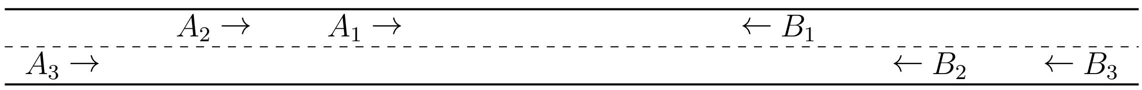
**Question 3.** What could be other sensible expansion rules for the city? State and answer questions analogous to Questions 1 and 2 for them. *[10 points]*

# Section C: Busy footbridge

Consider a long east-west footbridge that is just wide enough for two people to pass each other, and hence can be considered as having two lanes. In the morning, the bridge is initially empty. People arrive in a steady stream from both ends and join either lane randomly with equal probability.

When two people walking in opposite directions meet, they pass each other without incident if they are walking in different lanes. Otherwise there is a 'collision' and one of the two must change over to the other lane before they can pass each other and continue walking. In the absence of collisions, people remain in their lane when walking, and we assume that people walking in the same direction are sufficiently spread out that they do not overtake or interact with each other.

Let $A_m$ denote the $m^{\text{th}}$ person entering heading east and let $B_n$ denote the $n^{\text{th}}$ person entering heading west. An example of the lanes chosen by $A_1$, $A_2$, $A_3$ and $B_1$, $B_2$, $B_3$ when entering the footbridge is shown in the figure below. In this case, because $A_1$ is walking on their left and $B_1$ is walking on their right, the two will have a collision. In the collision, if $A_1$ moves over (and hence $B_1$ stays in their lane), then $A_1$ will collide with $B_2$ and $B_1$ will collide with $A_2$. If instead $A_1$ stays (and hence $B_1$ moves over), then $A_1$ will pass $B_2$ and $B_3$, and $B_1$ will pass $A_2$ but collide with $A_3$.




Let $p_{m,n}$ be the probability that $A_m$ and $B_n$ collide, assuming that they meet before either has left the bridge. Let $q_m$ denote the limiting value of $p_{m,n}$ as $n$ becomes large. (Throughout this question, you may assume that this limiting value exists.) Calculate $p_{1,n}$, $p_{2,n}$, $q_1$, $q_2$ and $q_3$ in the following cases.

**Question 1.** Each person heading east is either 'stubborn' (i.e. always stays in their lane) or 'polite' (i.e. always moves over in a collision) with equal probability. [*10 points*]

**Question 2.** When two people collide, the person heading east looks ahead and chooses to stay in their lane or move over in such a way as to maximise the time until their next collision. (Since everyone heading east is following this strategy, each of them can predict the outcome of every collision in front of them.) [*10 points*]

**Question 3.** When two people collide, a random one of them moves over, with equal probability. [*10 points*]

**Question 4.** What other strategies might people use to resolve a collision? How can you make the model more realistic? How do the collision probabilities change? [*10 points*]