Singapore International Mathematics Challenge 2016

Experior. Expono. Excedo.

THEME: COMMUNICATION

Communication is at the heart of the modern age. Historically it concerned face-to-face interactions, but as time has evolved the notion of communication at a distance has become more and more important, starting with the telegraph, the telephone and radio and passing on to modern notions like fibre-optic cables and bouncing radio waves off fixed bodies in space. With the advent of the internet, computer communication networks have assumed a central role.

The aim of the SIMC 2016 is to explore some of these aspects. There is a problem about the layout of computer networks: how resilient is a particular network, and how can we speed up communication in the network? There is a problem about satellites, examining how they should be placed for good communication to be possible. And there is a problem about imperfect communication: a rendezvous problem where one participant cannot communicate his position to the other participant. Each of these problems focuses on one feature of modern communication.

About the Challenge:

There are three parts to the Challenge – Part A, B and C. The three parts carry equal mark with each part consists of 3 to 4 questions. The weighting for each question indicates its contribution to the final score. It also serves as a guide on the amount of time and effort your team should spend on the question. Different teams may find different parts easy and hard. Each part also has an open-ended section, where you are invited not to solve a specific question but rather to invent new models and analyse them.

Written Report Requirements:

1. Your report should not exceed 10 pages. It will be read by judges. Present the main results and ideas only. You may omit details such as the steps of computations. However, you should be able to explain the details of your work during the oral presentations, if requested by the judges.

2. Teams should acknowledge all sources used.

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3. Your report must:
   - Be saved in PDF format.
   - Be single spacing with font size 12. Do not try to squeeze in the details by reducing the spacing between lines, or the margins.
   - Include your school name in FULL, on the first page of your report.

Oral Presentations:

1. Each team has to present their solutions 3 times – each presentation covers all three parts of the challenge problems (Part A, B and C).
2. Each presentation lasts 20 minutes and will be followed by a 5-minute Question & Answer session.

There are 6 printed pages (excluding the pink cover page)

Part A: Efficient Networks

In this question we imagine a large network of computers, linked by cables. Initially, we have a network of n^2 computers, arranged in a square grid. Neighbouring computers are joined by a cable, as shown in Figure 1 (left).

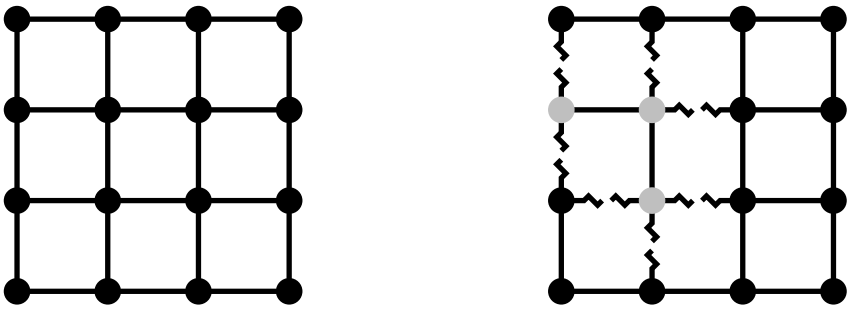

Figure 1: Example network with $n = 4$. Left: The original network. Right: The breaking of 7 connections has disconnected a set of 3 computers from the rest of the network.

If say two cables break from a corner computer, then clearly that computer is cut off from the rest of the network. We are concerned with some cables breaking so that some k of the computers all become unable to communicate with the remaining $n^2 - k$. For example, if 7 cables break as shown in Figure 1 (right) then $k = 3$ computers become disconnected from the remaining 13.

Question 1a. If $n = 100$, what is the smallest number of cables whose failure would disconnect some set of $k = 400$ computers from the rest?

Question 1b. If $n = 100$, what is the smallest number of cables whose failure would disconnect some set of $k = 3600$ computers from the rest?

[*10 points*]

Suppose now that our concern is to minimise transmission times. For example, sending a message from A to B in Figure 2 would need 4 cables to be traversed (in other words, the 'distance apart' of A and B is 4), which we assume takes 4 units of time. We have the option of adding some 'superhighways': a superhighway is a connection between two computers, using a very fast cable, so fast that the time needed for a signal to traverse it is negligible. For example, if we add a superhighway between C and D in Figure 2 then the time needed for a message to go from A to B is down to 3 – so the 'distance' between them is now down to 3.

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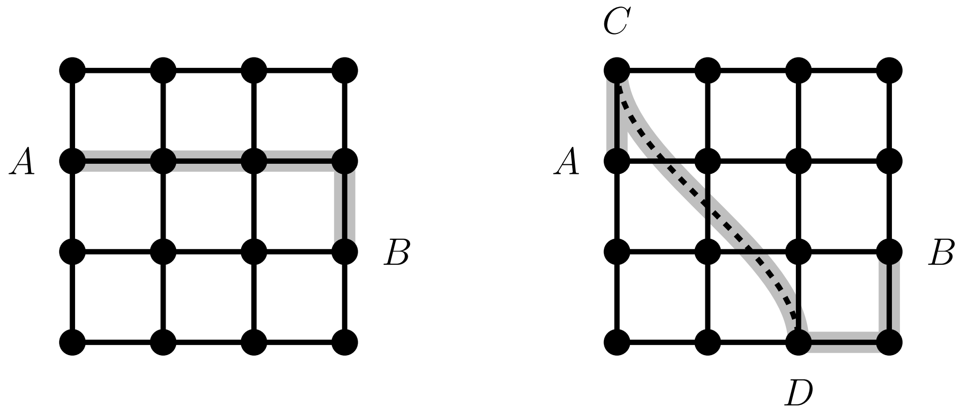

Figure 2: Example network with $n = 4$. Left: The distance between A and B in the original network is 4. The highlighted path is one of several length-4 paths between A and B. Right: After the superhighway between C and D is added, the distance between A and B is reduced to 3. The highlighted path is one of two length-3 paths between A and B.

Question 2a. If we are only allowed to add one superhighway, where should we place it so as to minimise the maximum distance that any signal has to travel in the resulting network? Assume for simplicity that n is odd.

Question 2b. If we are only allowed to add two superhighways, where should we place them so as to minimise the maximum distance that any signal has to travel in the resulting network? Again, assume that n is odd.

[*10 points*]

Question 3a. What is the smallest number of superhighways we can add in such a way that *every* pair of computers are now at distance at most 2? Give an approximate answer ignoring lower-order terms. (So for example if the exact value is $3n^3 + 6n^2 + 7$ then an answer of $3n^3$ is fine.)

Question 3b. What is the smallest number of superhighways we can add in such a way that *every* pair of computers are now at distance at most 4? Again, an approximate answer is sufficient.

[*10 points*]

Question 4. Investigate other natural/sensible/realistic starting networks instead of a square grid. What observations can you make? Try to justify your claims.

[*10 points*]

Part B: Satellites

For the purposes of this question, assume that the Earth is a sphere of fixed radius equal to 1 (see Figure 3 below). Hence, the "surface distance" (by which we mean distance as measured along the Earth's surface) between two points on the Earth is equal to the angle (measured in radians) between the two corresponding rays from the centre of the Earth. We wish to place some satellites in orbit, all at the same constant altitude h (which we are free to choose). For simplicity, we ignore the orbital motion of the satellites and the rotation of the Earth, i.e. the satellites can be treated as fixed points above a fixed Earth.

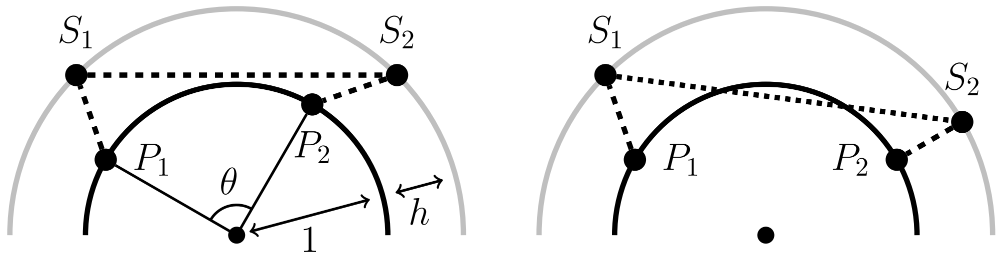

Figure 3: Satellites S_1 and S_2 are placed at altitude h above the Earth. Persons P_1, P_2 can communicate directly with each corresponding satellite as indicated by the dashed lines. Left: P_1 and P_2 (who are separated by a surface distance θ) can communicate by relaying a signal via S_1 and S_2. Right: P_1 and P_2 can not communicate since the signal path (dotted line) is blocked by the Earth.

Radio signals travel in a straight line, so a person on the surface of the Earth can communicate directly with a satellite in orbit only if it is above the person's horizon, i.e. if the line of sight between the two does not intersect the interior of the Earth. Similarly, two satellites can communicate directly only if the line between them does not intersect the interior of the Earth (but for simplicity we assume the two satellites can communicate if the line between them is tangent to the Earth).

Question 1. Consider a single satellite at altitude h above the North Pole. How far (along the Earth's surface) can a person be from the North Pole while being in range of this satellite? What is the surface area of the Earth covered by the satellite? (You may use the famous result by Archimedes that the area of the curved surface of a spherical cap on a unit sphere is given by $A = 2\pi d$ where d is the height of the cap.) Verify that, for very small h, the maximal surface distance is approximately $\sqrt{2h}$ and the area approximately $2\pi h$. What are the approximate values of the maximal surface distance and area when h is very large?

[*5 points*]

Question 2. Suppose that the cost of placing a satellite in orbit at altitude h is equal to its orbital energy $1 - 1/(2(1 + h))$. Determine the cheapest altitude and placement of a constellation of satellites that together cover the entire Earth. (Recall that we are ignoring the orbital motion of the satellites.)

[*15 points*]

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Question 3a. Suppose that radio signals travel at speed c (the speed of light) and that each satellite used in delivering a message adds a processing delay time p between reception and retransmission of a signal. Thus, the transmission time of a message depends on the distance travelled by the signal and the number of satellites used in relaying it. Assuming you have an unlimited supply of satellites to cover the Earth with, what is the optimum altitude h to place the satellites at to minimise the worst (longest) transmission time? (You may need to make use of some reasonable approximations.)

[*10 points*]

Question 3b. Investigate what would happen if we minimise not the *worst* transmission time, but some other sensible quantity.

[*10 points*]

Part C: Remote Island

Ficticia is a remote island in the Pacific ocean, surrounded by thousands of square kilometres of open sea. Although travelling to and from Ficticia is difficult, the island attracts many tourists, who come to fish in the rich waters just off the coast of Ficticia.

Mr Robinson was one such tourist, rowing in the waters by the island, when a thick fog came down on the region. The fog completely blocked Mr Robinson's sight of the island and he lost track of its location. Realising that finding his way back to the island would be impossible, he radioed the rescue workers on the island for help. The rescue workers picked up the distress signal and used its strength and direction to determine the exact location of Mr Robinson. They sent a rescue boat to pick him up.

Staying off shore at night unprepared would be extremely dangerous, so Mr Robinson wishes to be rescued before sunset. He can either stay in one place, or he can keep rowing. Rowing at his top speed, Mr Robinson can travel $V = 10$ kilometres before sunset. The rescue boat has the same top speed. For simplicity, suppose that Mr Robinson's boat and the rescue boat are points and that Mr Robinson is rescued when these two points meet. Mr Robinson has a compass, so he always knows in which direction he is going. However, the only information that he has regarding his location is that initially he is not more than $R = 40$ kilometres away from the island. Any location within radius R from the island is equally likely. (During his subsequent rowing, he might end up more than a distance R from the island.)

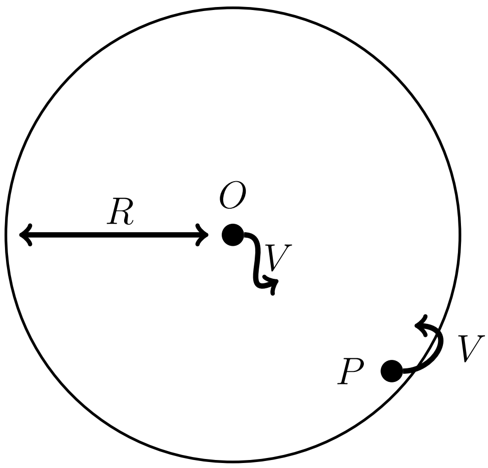

Figure 4: Mr Robinson is lost at position P within a radius $R = 40\,\text{km}$ of Ficticia at position O. He can row a distance $V = 10\,\text{km}$ before sunset, and must meet the rescue boat sent from Ficticia, travelling at the same speed, within that time.

Question 1. Suppose that the rescue workers respond to Mr Robinson's distress call and inform him about his exact location. Mr Robinson and the rescue boat constantly keep in contact. In this situation, of course, Mr Robinson and the rescue boat will head towards each other as fast as they can. What is the probability that Mr Robinson will be rescued before sunset?

[*5 points*]

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For the questions below, assume instead that Mr Robinson's radio is damaged: it can send, but not receive, transmissions. Hence the rescue workers know the exact location of Mr Robinson, but they cannot transmit this information to him. On the other hand, Mr Robinson can let the rescue workers know what trajectory he is going to follow. For example, he might decide to head north for 10 km, or he might decide to head south for 5 km and then east for 5 km.

Question 2. Determine (with proof) a strategy for Mr Robinson that maximises his chance of being rescued by sunset. Consider the two cases (i) $R = 40\,\text{km}$ and (ii) $R = 15\,\text{km}$.

[*10 points*]

Question 3. Suppose that the rescue boat has only *half the speed* of Mr Robinson's boat, so it can travel only 5 km before sunset. Determine (with proof) a best strategy for Mr Robinson in the two cases (i) $R = 40\,\text{km}$ and (ii) $R = 10\,\text{km}$.

[*15 points*]

Question 4. What do you think could be a more realistic model to simulate the location of a boat that got lost in a fog in the waters close to a remote island? Also, what could be realistic scenarios regarding the type of communication that the boat can have with the island? Carefully explaining what assumptions you are choosing to make, find a sensible rescue strategy for any scenario that you have described.

[*10 points*]