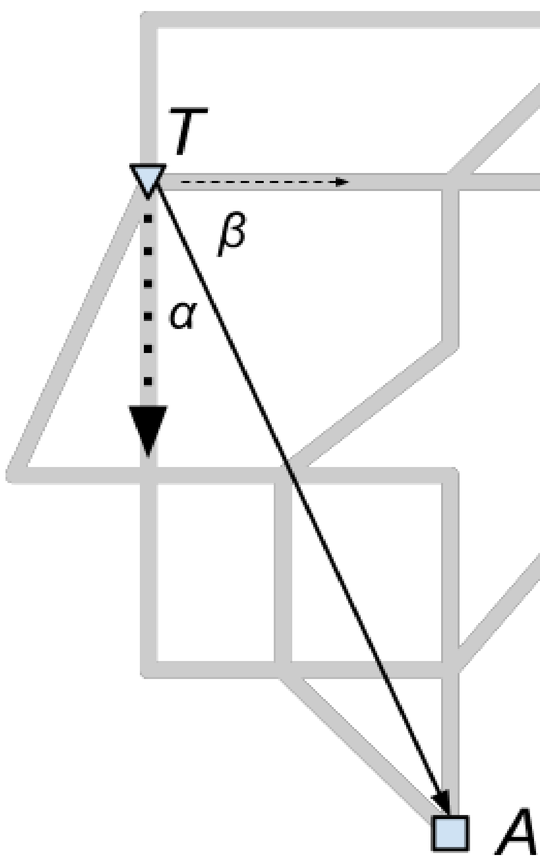
Challenge 1

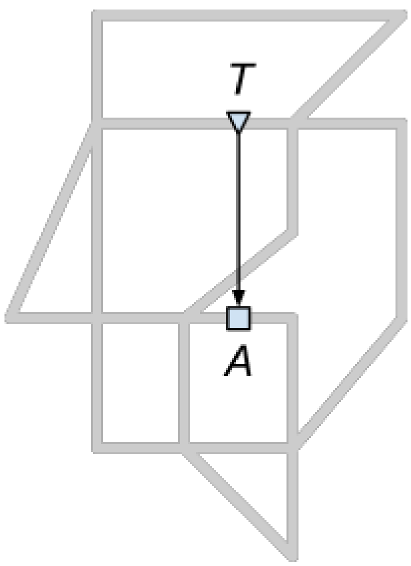
Little Amy is lost in a maze full of monsters and her dog Tigernal is trying to find his way to the owner. The dog doesn't see Amy (it's a maze) but can sniff through walls (you can think of a hedge maze). Thanks to his sense of smell, Tigernal knows the direction towards Amy at all times and is always moving along the maze's corridors trying to reduce his distance (along a straight line) to Amy as quickly as possible so that the angle between his movement and the direction towards the girl is as small as possible. At the same time, Amy is sitting still and waiting for her dog to save her.

From the mathematical point of view, the maze is a graph *G* in the plane and with all edges being line segments. Amy's position is some fixed point *A* in *G* while Tigernal's position is a point *T* moving along edges of *G*. Here is an example:



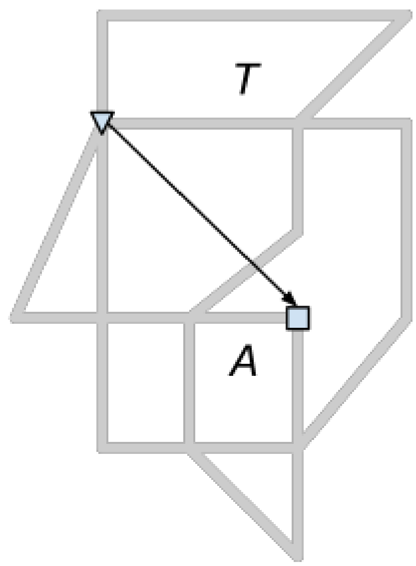
In this situation the dog will proceed downwards (arrow in the dotted line) because $\alpha < \beta$.

Unfortunately, sometimes the dog cannot find the girl. It happens in two cases:



Case 1

In a middle of a passage, if the direction towards Amy is perpendicular to the passage, then Tigernal cannot proceed because choosing either way would increase the distance to Amy.



Case 2

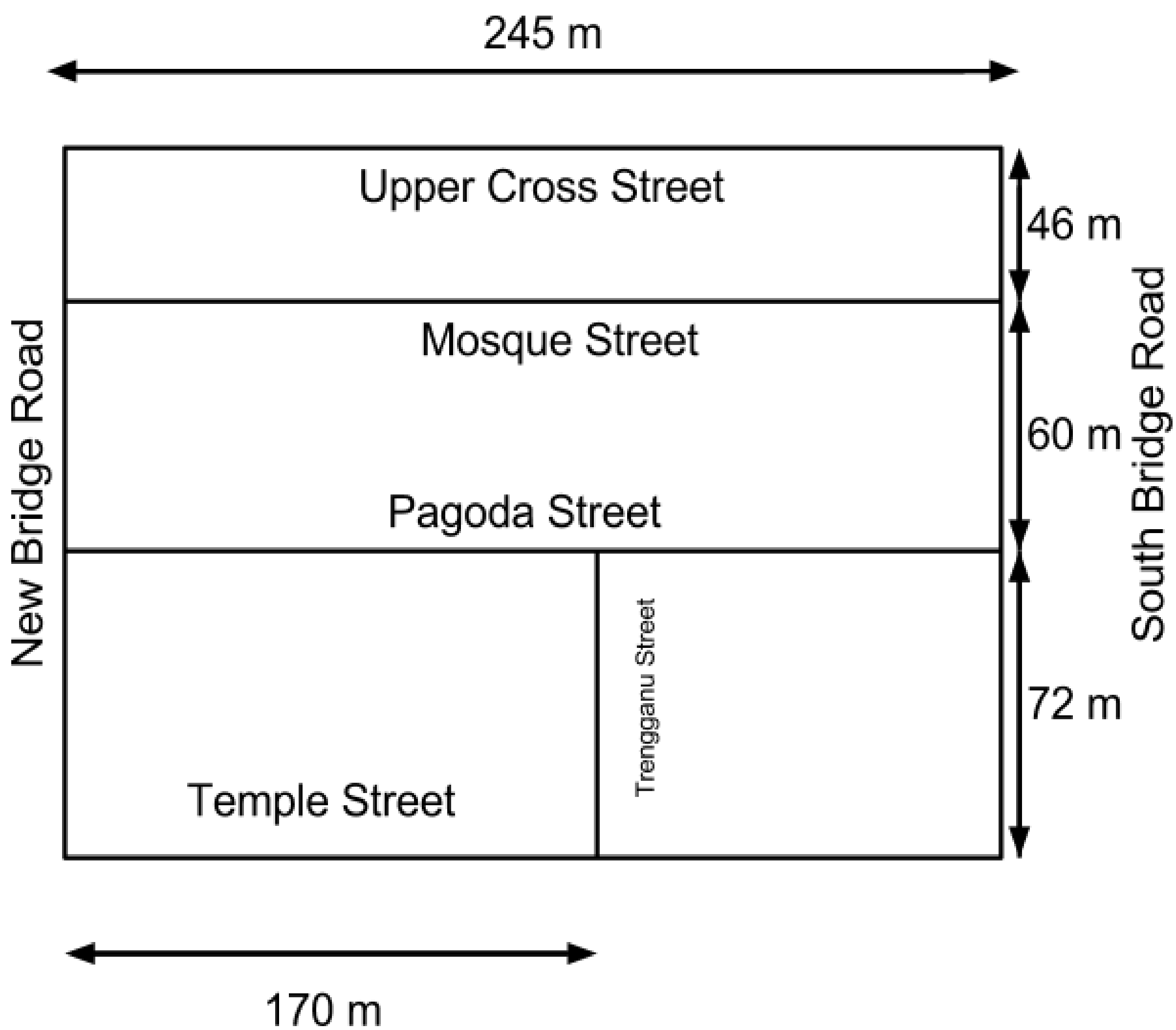
At a junction, if two smallest angles between the passages and the direction towards *A* are equal, then Tigernal doesn't know how to proceed and is stuck

Now we assume that the initial positions of Amy and Tigernal are random throughout the whole labyrinth (might be in at junction or might be in a middle of a passage) because the girl and the dog were thrown into the maze by an evil magic. Will the dog find his owner or are they doomed to be forever stuck there?

Example 1. If the maze is a unit square, then whenever Amy and Tigernal are initially on the opposite sides, then the dog cannot find his way to the girl. However, whenever they are initially on the same side or on adjacent sides, then they will eventually meet. Thus the probability that they are stuck forever in the labyrinth is $\frac{1}{4}$ and the probability that they will meet eventually is $\frac{3}{4}$.

Problems

a) The diagram below shows the schematic map of Chinatown area (see Annex A for the actual map). Find the probability that the dog and the girl will meet if their initial positions are random locations on this map (in the hypothetical situation of an apocalyptic event that turned Singapore into a maze full of monsters).



b) Given an arbitrary map, what are the conditions the initial position of the dog and the girl must satisfy so that they meet eventually?

c) Given an arbitrary map, propose a way of how to find the probability that Tigernal and Amy will meet and the expected length of the path from Tigernal to Amy.

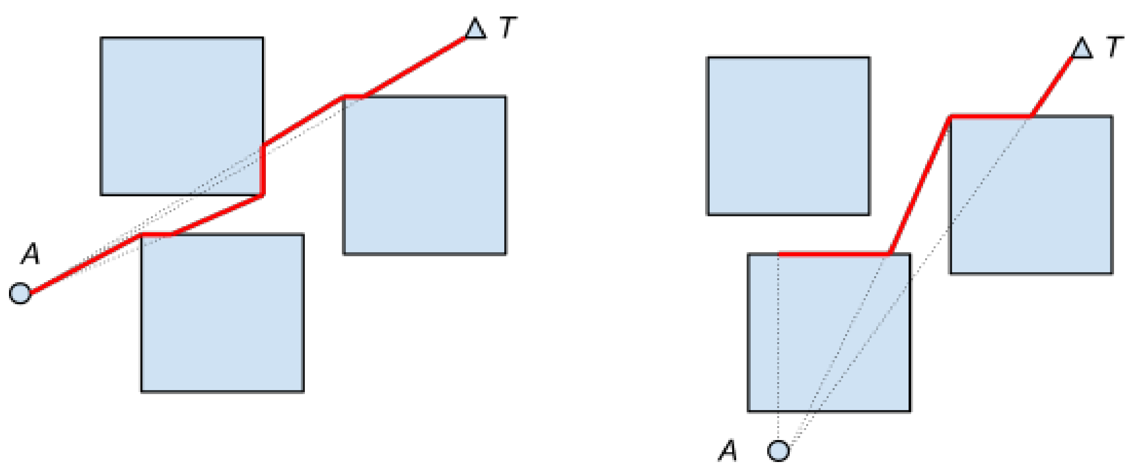
Challenge 2

Little Amy is lost in an enchanted town full of monsters and her dog Tigernal is trying to find his way to the owner. All buildings in the town are unit squares with sides parallel to the coordinate grid. Buildings have no walls or windows and cannot be entered.

Again, thanks to his keen sense of smell, Tigernal knows the direction towards Amy at all times and is always moving making the angle between his direction of motion and the direction towards the girl as small as possible. In other words, he is running along a straight line until he hits the wall of a building and then goes along the wall. At the same time, Amy is sitting still and waiting for her dog to escape her. Let *A* be Amy's position. Notice that the dog doesn't see (or doesn't really care about) a building until he hits the wall.

Unfortunately, if the dog is at the wall and the direction towards the girl is perpendicular to the wall, then Tigernal cannot proceed. They are doomed to be stuck forever.

Example 2. A situation when the dog and the girl meet (left diagram) and a situation when they don't (right diagram) with the trajectory of the dog's movement in each case:



Throughout Challenge 2, we assume that Amy's position is fixed and that the map of the town is not changing. Also, there are finitely many square buildings and distances between buildings are positive (so buildings cannot touch each other).

The probability P that Tigernal finds Amy is defined as follows. Let $r > 0$. Tigernal's initial position is a random point in the circle of radius *r* centred at *A* - Amy's position. Let $A(r)$ be the area of the set of all points *T* such that Tigernal will eventually meet Amy if he starts at *T*. Then we define P to be the limit of the ratio $\frac{A(r)}{\pi r^2}$ as r approaches infinity.

Problems
Recall that all buildings in the town are unit squares with sides parallel to the coordinate grid,

a) Suppose that there is only one building. Depending on *A*, describe all points *T* such that Tigernal will find Amy if he starts at *T*. Find the probability, P that the girl and the dog will eventually meet if there is only one building (P should depend on the position of the girl relative to the building).

b) Does there exist a configuration of finitely many buildings such that $P = 0$?
If the answer is yes, justify your answer and give the minimum number of buildings in a configuration with $P = 0$. If the answer is no, prove it.

c) Suppose that there is an additional restriction that the minimum distance between buildings cannot be smaller than some positive constant *E*. What is the maximum value of *E* such that it is still possible to find a finite configuration of buildings with $P = 0$?

Challenge 3

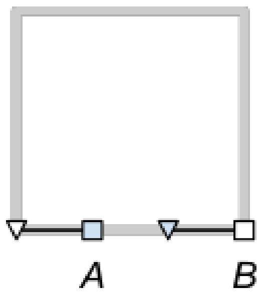
Now two little kids - Alan and Bob are lost in a maze full of monsters. This time they have their dogs with them. However, the dogs have been switched and now Alan is holding Bob's dog on a leash while Bob is holding Alan's dog on another leash. The kids are too scared to be conscious of their movement and are just being pulled by the dogs. Each dog smells its master and moves towards him, at the same time pulling the other master on the leash. The dogs' movement is described by the same laws as in Challenge 1. However, the owners are no longer stationary.

The assumptions are

1. The speeds of the two dogs are equal.
2. The length of each leash is a fixed positive number *L*. It means that at all times, the distance (along the corridors) between the dog and the opposite owner doesn't exceed *L*.
3. The movement in each mismatched couple is driven by the dog. Once the distance between the dog and the kid who is holding the leash approaches *L*, the kid starts moving in the same direction as the dog.
4. A happy ending happens when at least one of the owners meets his dog.

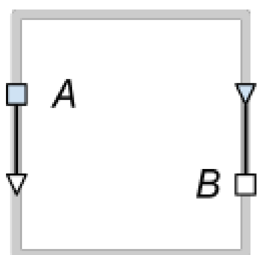
Here are a few examples of initial positions and movements of the dogs and the owners in a simple square maze:

Example 3



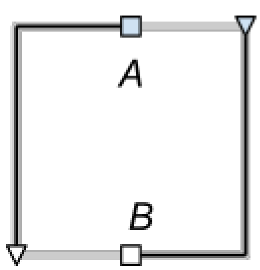
The initial positions of the two masters are shown by little squares and the dogs by little triangles. Here, Alan's dog will be moving towards Alan (in the left direction) pulling Bob along. At the same time, Bob's dog will be moving towards Bob (in the right direction), but Alan will remain still because the leash length allows to. When Alan's dog reaches Alan, Bob's dog will be at the same position (provided the distances between the four points are equal). Happy ending.

Example 4



Here both dogs are stuck - they cannot move because in both cases the direction towards the master is perpendicular to the corridor they are in. No happy ending for them.

Example 5



If the leash is sufficiently long ($\frac{3}{2}$ side lengths of the square on the illustration), then it might happen that the kids and the dogs will end up circling around forever. Here, both dogs are trying to pull the kids in the counter-clockwise direction. Obviously, the distance between each dog and his owner is not changing - it remains to be $\frac{1}{2}$ side length of the square. Again, no happy ending.

Problem

State and solve as many scenarios/models/questions as you can. Your theorem(s) may or may not be similar to those you came up with in Challenge 1 and Challenge 2.

Challenge 4

Problem

Suggest and provide real life applications where the models (Challenge 1 to 3) can be broadly applied.